Negative-energy perturbations in general axisymmetric and helical Maxwell-Vlasov equilibria

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(Received 26 December 1996)

The expression for the free energy of arbitrary perturbations of general Vlasov-Maxwell equilibria derived by Morrison and Pfirsch is transformed and put in a concise form, which is subsequently evaluated for arbitrary equilibria which have one ignorable coordinate, e.g., axisymmetric and helical equilibria, in the case of internal perturbations, i.e., perturbations which vanish outside the plasma, and on its boundary. In order to generate the electric currents necessary for equilibrium in the presence of pressure gradients, the equilibrium distribution function of at least one particle species must be anisotropic. As a consequence, these equilibria always allow negative-energy perturbations, *without requiring a large spatial variation of the perturbation across the equilibrium magnetic field*. [S1063-651X(97)06605-1]

PACS number(s): 52.35.Mw

I. INTRODUCTION

The existence of negative-energy perturbations in an otherwise stable, collisionless plasma could lead to instabilities in the presence of dissipation, or it could lead to nonlinear instabilities (and thus cause anomalous transport) through nonlinear coupling with perturbations of positive energy [1–4]. Therefore, it is of paramount importance to investigate under what conditions a given plasma equilibrium configuration admits negative-energy perturbations. Considering arbitrary perturbations of general Vlasov-Maxwell equilibria, Morrison and Pfirsch [5,6] derived expressions for the second variation of the free energy, and concluded that negative-energy modes exist in any Maxwell-Vlasov equilibrium whenever the unperturbed distribution function $f_\nu^{(0)}$ of any particle species ν deviates from monotonicity in v^2 and/or isotropy in the vicinity of a single point, i.e., whenever the condition $(\mathbf{v}\cdot\mathbf{k})[\mathbf{k}\cdot(\partial f_\nu^{(0)}/\partial v)]>0$ holds (in the frame of reference of minimum equilibrium energy) for any particle species ν for some position vector $\mathbf{x}$ and velocity $\mathbf{v}$ and for some local wave vector $\mathbf{k}$. The proof of this result was based on infinitely strongly localized perturbations, which correspond to $|\mathbf{k}|\to\infty$. *This raises the question of the degree of localization actually reguired for negative-energy modes to exist in a certain equilibrium*. Studying Maxwell-Vlasov plasma configurations, in which the equilibrium quantities depend only on *one* spatial coordinate, Correa-Restrepo and Pfirsch [7–9] showed that negative-energy modes exist for any deviation of the equilibrium distribution function of any of the species from monotonicity and/or isotropy, *without having to impose any restricting conditions on the perpendicular wave number* $k_\perp$, i.e., without requiring large $k_\perp$. Detailed investigations of negative-energy perturbations in plane and circularly symmetric plasmas have also been done within the framework of Maxwell-drift kinetic theory by Throumoulopoulos and Pfirsch [10,11]. Within the framework of Maxwell-Vlasov theory, the results obtained for one-dimensional configurations were later shown to be valid also for a class of equilibria which depended not only on one, but on *two* spatial coordinates. The equilibria considered were, however, restricted in the sense that they had only ''toroidal'' equilibrium currents, i.e., currents flowing in the direction of the ignorable coordinate, e.g., the toroidal angle φ in axisymmetry, and were thus of the $\beta_p=1$ type [12].

In the present paper, the results obtained for $\beta_p=1$ equilibria are extended to the considerably more interesting case of general symmetric equilibria with one ignorable coordinate, e.g., axisymmetric tokamaks and helical configurations, which have both ''toroidal'' and ''poloidal'' currents. These investigations make extensive use of the Poisson bracket formalism.

In order to generate the currents necessary for a general axisymmetric or helical equilibrium in the presence of pressure gradients, the equilibrium distribution function of at least one particle species must depend not only on the particle energy $\mathcal{H}_\nu$, but also on the canonical momentum $\mathcal{P}_{\nu 3}$ in the toroidal direction, which is the momentum canonically conjugated to the ignorable coordinate (e.g., the toroidal angle φ in a tokamak), and on at least one of the other two independent constants of the motion. Because of this, the configurations always allow negative-energy perturbations. It is shown that large spatial variations (i.e., short wavelengths) of the perturbations across the equilibrium magnetic field are not required, a feature which could enhance the importance of this kind of perturbations in helical and axisymmetric configurations.

In Sec. II, the expression for the free energy $\delta^2 H$ available upon arbitrary perturbations of general Maxwell-Vlasov equilibria derived by Morrison and Pfirsch [5] is transformed and put in a concise form. Section III describes the geometry and properties of the equilibrium distribution functions of the configurations. The expression for the free energy is then evaluated in Sec. IV for general symmetric equilibria. Considering internal perturbations, i.e., those which vanish outside the plasma, and on its boundary, the minimizing perturbations are obtained in Sec. V, where the expression for the minimized energy is also obtained. This expression is then discussed in Sec. VI. The results are summarized in Sec. VII.

In Appendix A, useful relations concerning the Poisson brackets which are needed for the calculations are derived. Finally, in Appendix B, solutions of the Euler equation derived in Sec. V are found.

II. PERTURBATION ENERGY FOR GENERAL MAXWELL-VLASOV EQUILIBRIA

The expression for the free energy $\delta^2 H$ available upon arbitrary perturbations of general Maxwell-Vlasov equilibria derived by Morrison and Pfirsch [5,6] assumes a particularly simple form when it is evaluated for the case that the initial perturbation $\delta\mathbf{B}_{t=t_0}=\nabla\times\delta\mathbf{A}_{t=t_0}$ of the magnetic field vanishes. $\delta\mathbf{A}_{t=t_0}=\mathbf{0}$ can be chosen independently of the generating functions G_ν for the particle position and velocity perturbation (see Ref. [6]) because Maxwell's equations allow for the production of a displacement current that makes a given particle-field configuration consistent. Also, for the perturbations considered here, it is possible to show that the initial particle electric current density perturbation $\delta\mathbf{j}_{t=t_0}$ can be made at least arbitrarily small. According to Ref. [9], Eq. (9), one then obtains

$$\delta^2 H=\sum_\nu \int \frac{d^3x d^3v}{2m_\nu}\left\{(d_\nu G_\nu)\left(\mathbf{F}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial\mathbf{v}}-\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\cdot\frac{\partial G_\nu}{\partial\mathbf{x}}\right)\right\} + \frac{1}{8\pi}\int d^3x\ \delta E^2. \tag{1}$$

Here $\mathbf{x}$ and $\mathbf{v}$ are space and velocity coordinates, respectively. $f_\nu^{(0)}(\mathbf{x},\mathbf{v})$ is the equilibrium distribution function for particles of species ν, which have mass m_ν and electric charge e_ν. $G_\nu(\mathbf{x},\mathbf{v})$ is the arbitrary generating function for the perturbations $\delta\mathbf{x}$ and $\delta\mathbf{v}$ of the particle position and velocity, respectively [5]. $\delta E^2/(8\pi)$ is the perturbation in the electric-field energy density, and $d_\nu=[d/dl]_{\text{along unperturbed orbits}}$ is the equilibrium Vlasov operator, i.e.,

$$d_\nu=\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\mathbf{a}_\nu^{(0)}\cdot\frac{\partial}{\partial\mathbf{v}}, \tag{2}$$

where

$$\mathbf{a}_\nu^{(0)}=\frac{e_\nu}{m_\nu}\left(\mathbf{E}^{(0)}+\frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right), \tag{3}$$

with $\mathbf{E}^{(0)}=-\nabla\Phi^{(0)}$ and $\mathbf{B}^{(0)}=\nabla\times\mathbf{A}^{(0)}$ the time-independent equilibrium electric and magnetic fields, respectively, and

$$\mathbf{F}_\nu^{(0)}=\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}. \tag{4}$$

The Lagrangian L_ν of a particle of species ν is

$$L_\nu=\frac{m_\nu}{2}\mathbf{v}^2+\frac{e_\nu}{c}\mathbf{A}^{(0)}(\mathbf{x})\cdot\mathbf{v}-e_\nu\Phi^{(0)}(\mathbf{x}), \tag{5}$$

from which the momentum canonically conjugated to $\mathbf{x}$ follows:

$$\mathbf{P}_\nu=\frac{\partial L_\nu}{\partial v}=m_\nu\mathbf{v}+\frac{e_\nu}{c}\mathbf{A}^{(0)}. \tag{6}$$

Taking into account the relations derived in Appendix A, Eqs. (A9) and (A12), and making use of Poisson brackets, which for any two functions $f_1[\mathbf{x},\mathbf{v}=(p/m_\nu)-(e_\nu/m_\nu c)\mathbf{A}^{(0)}(\mathbf{x})]$ and $f_2(\mathbf{x},\mathbf{v}=\cdots)$ are defined by the equation

$$[f_1,f_2]=\left.\frac{\partial f_1}{\partial\mathbf{x}}\right|_{\mathbf{p}}\cdot\left.\frac{\partial f_2}{\partial\mathbf{p}}\right|_{\mathbf{x}}-\left.\frac{\partial f_1}{\partial\mathbf{p}}\right|_{\mathbf{x}}\cdot\left.\frac{\partial f_2}{\partial\mathbf{x}}\right|_{\mathbf{p}}, \tag{7}$$

one can write

$$d_\nu G_\nu=\left[\frac{dG_\nu}{dt}\right]_{\text{along unperturbed orbits}}=[G_\nu,H_\nu], \tag{8}$$

where H_ν is the unperturbed Hamiltonian, i.e.,

$$H_\nu=\frac{1}{2m_\nu}\left[p_\nu-\frac{e_\nu}{c}\mathbf{A}^{(0)}(\mathbf{x})\right]^2+e_\nu\mathbf{\Phi}^{(0)}(\mathbf{x}). \tag{9}$$

the expression for the perturbation energy, Eq. (1), can then be written as

$$\delta^2 H=\sum_\nu\int\frac{d^3x\ d^3p}{2m_\nu^3}[G_\nu,H_\nu][f_\nu^{(0)},G_\nu] + \frac{1}{8\pi}\int d^3x\ \delta E^2, \tag{10}$$

Here, $\mathbf{x},\mathbf{p}$, instead of $\mathbf{x},\mathbf{v}$, are now taken as the independent variables. Equation (10) is esentially Eq. (13) of Ref. [6] evaluated for perturbation with $\delta\mathbf{A}(\mathbf{x},t=0)=0$.

III. EQUILIBRIUM

In generalized coordinates q_i, $i=1,\ldots,3$, symmetric configurations are now considered which do not depend on q_3, but only on q_1 and q_2. Examples of these are axisymmetric equilibria, which do not depend on the toroidal angle φ, and helically symmetric equilibria, which, in cylindrical coordinates r,φ,z, depend only on r and $u\equiv m\varphi+lz$, but not explicitly on z, with m and l arbitrary integer and real numbers, respectively. The equilibrium magnetic field $\mathbf{B}^{(0)}=\nabla\times\mathbf{A}^{(0)}$ can then be expressed in the general form

$$\mathbf{B}^{(0)}=\nabla\times(A_i^{(0)}(q_1,q_2)\nabla q_i) = \frac{1}{J(q_1,q_2)}\left\{\frac{\partial A_3^{(0)}}{\partial q_2}\frac{\partial\mathbf{x}}{\partial q_1}-\frac{\partial A_3^{(0)}}{\partial q_1}\frac{\partial x}{\partial q_2}\left(\frac{\partial A_2^{(0)}}{\partial q_1}-\frac{\partial A_1^{(0)}}{\partial q_2}\right)\frac{\partial\mathbf{x}}{\partial q_3}\right\}, \tag{11}$$

where

$$J(q_1,q_2)=\frac{\partial\mathbf{x}}{\partial q_1}\cdot\frac{\partial\mathbf{x}}{\partial q_2}\times\frac{\partial\mathbf{x}}{\partial q_3}=\frac{1}{\nabla q_1\cdot\nabla q_2\times\nabla q_3}. \tag{12}$$

Since the equilibrium fields $\mathbf{E}^{(0)}=-\nabla\Phi^{(0)}$ and $\mathbf{B}^{(0)}=\nabla\times\mathbf{A}^{(0)}$ are time-independent, the particle energy

$$\mathcal{H}_\nu=\frac{m_\nu}{2}\,\mathbf{v}^2+e_\nu\Phi^{(0)}=\frac{1}{2m_\nu}\left[\mathbf{p}_\nu-\frac{e_\nu}{c}\,\mathbf{A}^{(0)}\right]^2+e_\nu\mathbf{\Phi}^{(0)} \tag{13}$$

is a constant of the motion, and since q_3 is an ignorable coordinate, the corresponding canonical momentum

$$\mathcal{P}_{\nu3}\equiv p_{\nu3}=m_\nu v_3+\frac{e_\nu}{c}\,A_3^{(0)}(q_1,q_2) \tag{14}$$

is also a constant of the motion (calligraphic letters are used here to denote constants of the particle motion). In the *five-dimensional* space (q_1,q_2,p_i), $i=1,\ldots,3$, the general equilibrium solution of Vlasov's equation

$$d_\nu f_\nu^{(0)}(q_1,q_2,p_i)=[f_\nu^{(0)},\mathcal{H}_\nu]=0 \tag{15}$$

is

$$f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{P}_{\nu3},\mathcal{K}_{\nu1},\mathcal{K}_{\nu2}), \tag{16}$$

where $\mathcal{K}_{\nu1}$ and $\mathcal{K}_{\nu2}$ are two further constants of the particle motion which are also q_3 independent. $\mathcal{K}_{\nu1}$ and $\mathcal{K}_{\nu2}$ are not explicitly known here, but it is assumed that they appear explicitly in the expression for $f_\nu^{(0)}$ in order to be able to construct *general* axisymmetric or helical equilibria. Exclusion of either $\mathcal{K}_{\nu1}$ or $\mathcal{K}_{\nu2}$, or both, from the expression for $f_\nu^{(0)}$ leads to *special* symmetric equilibria. One interesting example of these are the $\beta_p=1$ tokamaks, for which

$$f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{P}_{\nu3}). \tag{17}$$

Introducing a *local* Cartesian coordinate system with unit basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, such that $\mathbf{e}_3=(\partial\mathbf{x}/\partial q_3)/|\partial\mathbf{x}/\partial q_3|$, the velocity $\mathbf{v}$ at point $\mathbf{x}$ can be decomposed into three orthogonal components v_{c1}, v_{c2}, and v_{c3} and, therefore, $\mathcal{H}_\nu=(m_\nu/2)\Sigma_i v_{ci}^2+e_\nu\Phi^{(0)}$, $\mathcal{P}_{\nu3}=[m_\nu v_{c3}+(e_\nu/c)\mathbf{A}^{(0)}\cdot e_3][|\partial\mathbf{x}/\partial q_3|]$. The components v_{c1} and v_{c2} do not contribute to the mean velocity $\langle\mathbf{v}\rangle_\nu$ of species ν, since $f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{P}_{\nu3})$ is an even function of v_{c1} and v_{c2}. This yields $\langle\mathbf{v}\rangle_\nu=e_3(1/n_\nu)\int_\infty^\infty d^3v\; v_{c3}f_\nu^{(0)}$, and the current density is

$$\mathbf{j}^{(0)}=\sum_\nu e_\nu n_\nu\langle\mathbf{v}\rangle_\nu=\mathbf{e}_3\sum_\nu e_\nu\int_\infty^\infty d^3v\; v_{c3}f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{P}_{\nu3}). \tag{18}$$

Therefore, for that class of equilibria there is a current only in the direction corresponding to the ignorable coordinate (the toroidal angle φ in axissymmetry) and the equilibria are of the $\beta_p=1$ type [12].

IV. PERTURBATION ENERGY FOR GENERAL SYMMETRIC EQUILIBRIA

Here, all physical quantities are periodic in the coordinate q_3 with, say, period 2π in axial symmetry or $2\pi/L_z$ in helical symmetry. Since the equilibrium does not depend on q_3, single modes corresponding to this coordinate can be considered. An appropriate ansatz for the generating function G_ν of the perturbations is then

$$G_\nu(\mathbf{x},\mathbf{p})=\tfrac{1}{2}\Psi_\nu(q_1,q_2,p_i)\{e^{i[k_3q_3+\Gamma_\nu(q_1,q_2,p_i)]}+\text{c.c.}\}, \tag{19}$$

$i=1,\ldots,3$, where Ψ_ν and Γ_ν are arbitrary *real* functions such that G_ν is a single-valued function of q_1, q_2 and of the $p_i's$. With this ansatz, one obtains

$$[G_\nu,H_\nu]=\frac{\partial G_\nu}{\partial\Psi_\nu}[\Psi_\nu,H_\nu]+\frac{\partial G_\nu}{\partial\Gamma_\nu}[\Gamma_\nu,H_\nu]+\frac{\partial G_\nu}{\partial q_3}[q_3,H_\nu], \tag{20}$$

$$[f_\nu^{(0)},G_\nu]=\frac{\partial G_\nu}{\partial\Psi_\nu}[f_\nu^{(0)},\Psi_\nu]+\frac{\partial G_\nu}{\partial\Gamma_\nu}[f_\nu^{(0)},\Gamma_\nu]+\frac{\partial G_\nu}{\partial q_3}[f_\nu^{(0)},q_3], \tag{21}$$

$$\frac{\partial G_\nu}{\partial\Psi_\nu}=\tfrac{1}{2}\{e^{i(k_3q_3+\Gamma_\nu)}+\text{c.c.}\}, \tag{22}$$

$$\frac{\partial G_\nu}{\partial\Gamma_\nu}=\frac{i}{2}\,\Psi_\nu\{e^{i(k_3q_3+\Gamma_\nu)}-e^{-i(k_3q_3+\Gamma_\nu)}\},\qquad \frac{\partial G_\nu}{\partial q_3}=k_3\,\frac{\partial G_\nu}{\partial\Gamma_\nu}. \tag{23}$$

Inserting Eqs. (20)–(23) in Eq. (10), and integrating with respect to q_3 between q_{30} and $q_{30}+2\pi/k_3$, yields

$$\delta^2H=\sum_\nu\int\frac{J(q_1,q_2)}{2m_\nu^3}\frac{\pi}{k_3}\,dq_1dq_2d^3p\{-[\Psi_\nu,H_\nu]\times[\Psi_\nu,f_\nu^{(0)}]-\Psi_\nu^2[g_\nu,H_\nu][g_\nu,f_\nu^{(0)}]\}+\frac{1}{8\pi}\int d^3x\;\delta E^2, \tag{24}$$

where

$$g_\nu\equiv k_3q_3+\Gamma_\nu(q_1,q_2,p_i), \tag{25}$$

and the Poisson bracket $[g_\nu,f_\nu^{(0)}]$ is given explicitly by

$$[g_\nu,f_\nu^{(0)}]=[g_\nu,\mathcal{H}_\nu]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}+k_3\frac{\partial f_\nu^{(0)}}{\partial\mathcal{P}_{\nu3}}+[g_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[g_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}. \tag{26}$$

V. EXTREMIZATION OF THE PERTURBATION ENERGY

As pointed out in Sec. II, the equilibrium Vlasov operator d_ν, given explicitly by Eq. (2), means differentiation with respect to time along the unperturbed orbits (see also Appendix A). Then, for any two functions $f_{1,2}(q_i,p_i)$, $i=1,\ldots,3$, the following relations are valid:

$$d_\nu[f_1,f_2]=\frac{d}{dt}[f_1,f_2]=[[f_1,f_2],\mathcal{H}_\nu]$$
$$=\left[\frac{df_1}{dt},f_2\right]+\left[f_1,\frac{df_2}{dt}\right]. \tag{27}$$

Owing to the lack of knowledge concerning the explicit form of the two constants of the motion $\mathcal{K}_{\nu1}$ and $\mathcal{K}_{\nu2}$, complete minimalization of $\delta^2 H$, Eq. (24), with respect to Γ_ν is not possible. Partial minimalization, however, can be accomplished if one imposes an appropriate constraint. This is done here by minimizing $\delta^2 H$ under the subsidiary condition that the functional $\delta^2 H_{\text{constraint}}$ remains unchanged, with

$$\delta^2 H_{\text{constraint}}=\sum_\nu \int \frac{J(q_1,q_2)}{2m_\nu^3}\frac{\pi}{k_3}dq_1dq_2d^3p\Bigg\{\Psi_\nu^2[g_\nu,\mathcal{H}_\nu]\times\left[[g_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[g_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\Bigg\}. \tag{28}$$

Accordingly, we minimize the auxiliary functional $\delta^2 H_{\text{aux}}$, defined by the relation

$$(\delta^2 H)_{\text{aux}}=\delta^2 H+\lambda(\delta^2 H)_{\text{constraint}}, \tag{29}$$

where λ is a Lagrange multiplier. The variation of $(\delta^2 H)_{\text{aux}}$ with respect to Γ_ν is

$$\delta_{\Gamma_\nu}(\delta^2 H)_{\text{aux}}=(\delta^2 H)_{\text{aux}}(\Gamma_\nu+\delta\Gamma_\nu)-(\delta^2 H)_{\text{aux}}(\Gamma_\nu)=\sum_\nu\int\frac{J(q_1,q_2)}{2m_\nu^3}\frac{\pi}{k_3}dq_1dq_2d^3p[-\Psi_\nu^2]\Bigg\{[g_\nu,H_\nu][\delta\Gamma_\nu,f_\nu^{(0)}]$$
$$+[\delta\Gamma_\nu,H_\nu][g_\nu,f_\nu^{(0)}]-\lambda[g_\nu,\mathcal{H}_\nu]\left[[\delta\Gamma_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[\delta\Gamma_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]$$
$$-\lambda[\delta\Gamma_\nu,\mathcal{H}_\nu]\left[[g_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[g_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\Bigg\}, \tag{30}$$

which, using the definition of the Poisson brackets, Eq. (7) and, in particular, Eq. (27), can be transformed to

$$\delta_{\Gamma_\nu}(\delta^2 H)_{\text{aux}}=\sum_\nu\int\frac{J(q_1,q_2)}{2m_\nu^3}\frac{\pi}{k_3}dq_1dq_2d^3p\Bigg\{\frac{\partial}{\partial\mathbf{p}}\cdot\left[\delta\Gamma_\nu\Psi_\nu^2(d_\nu g_\nu)\left[\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}\bigg|_{\mathbf{p}}-\lambda\left[\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}\frac{\partial\mathcal{K}_{\nu1}}{\partial\mathbf{x}}\bigg|_{\mathbf{p}}+\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\frac{\partial\mathcal{K}_{\nu2}}{\partial\mathbf{x}}\bigg|_{\mathbf{p}}\right]\right]\right]$$
$$-\frac{\partial}{\partial\mathbf{x}}\cdot\left[\delta\Gamma_\nu\Psi_\nu^2(d_\nu g_\nu)\left[\frac{\partial f_\nu^{(0)}}{\partial\mathbf{p}}\bigg|_{\mathbf{x}}-\lambda\left[\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}\frac{\partial\mathcal{K}_{\nu1}}{\partial\mathbf{p}}\bigg|_{\mathbf{x}}+\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\frac{\partial\mathcal{K}_{\nu2}}{\partial\mathbf{p}}\bigg|_{\mathbf{x}}\right]\right]\right]$$
$$-d_\nu\left[\left[\delta\Gamma_\nu\Psi_\nu^2\left[[g_\nu,f_\nu^{(0)}]-\lambda\left[[g_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[g_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\right]\right]\right]$$
$$+\delta\Gamma_\nu\left[d_\nu\left[\Psi_\nu^2[g_\nu,f_\nu^{(0)}]-\lambda\Psi_\nu^2\left[[g_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[g_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\right]\right.$$
$$\left.+[\Psi_\nu^2(d_\nu g_\nu),f_\nu^{(0)}]-\lambda\left[[\Psi_\nu^2(d_\nu g_\nu),\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[\Psi_\nu^2(d_\nu g_\nu),\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\right]\Bigg\}. \tag{31}$$

Here, $\delta\Gamma_\nu$ is taken to vanish outside the plasma, and on its boundary (i.e., internal perturbations are considered). Because of this, the term which is a divergence in **x** does not contribute. The term which is a divergence in **p** vanishes upon integration because $f_\nu^{(0)}\to 0$ for $\mathbf{p}\to\infty$. For the same reasons, the contribution of term $d_\nu(\delta\Gamma_\nu\Psi_\nu^2[g_\nu,f_\nu^{(0)}])$ also vanishes, as can be seen by taking into account the relations

$$d_\nu(\delta\Gamma_\nu\Psi_\nu^2[g_\nu,f_\nu^{(0)}])=[\delta\Gamma_\nu\Psi_\nu^2[g_\nu,f_\nu^{(0)}],H_\nu]$$
$$=(\partial/\partial\mathbf{x})\cdot[\delta\Gamma_\nu\Psi_\nu^2[g_\nu,f_\nu^{(0)}](\partial H/\partial\mathbf{p})]$$
$$-(\partial/\partial\mathbf{p})\cdot[\delta\Gamma_\nu\Psi_\nu^2[g_\nu,f_\nu^{(0)}]\times(\partial H_\nu/\partial\mathbf{x})].$$

In a similar way, it can be shown that the term

$$d_\nu\left[\lambda\,\delta\Gamma\,\nu\Psi_\nu^2\left[[g_\nu,\mathcal{K}_{\nu1}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[g_\nu,\mathcal{K}_{\nu1}]\,\frac{\partial f_{\boldsymbol{\nu}}^{(0)}}{\partial\mathcal{K}_{\nu1}}\right]\right]$$

does not contribute (for this to be valid, the functions $d_\nu g_\nu = [g_\nu,\mathcal{H}_\nu]$ and $[g_\nu,f_\nu^{(0)}]$ must be single valued. This is the case for the solutions found in Appendix B. Therefore

$$\begin{aligned}\delta_{\Gamma_\nu}(\delta^2H)_{\rm aux}=\sum_\nu\int\frac{J(q_1,q_2)}{2m_\nu^3}\,\frac{\pi}{k_3}\,dq_1dq_2d^3p(\delta\Gamma_\nu)\\ \times\Bigg\{d_\nu\left[\Psi_\nu^2[g_\nu,f_\nu^{(0)}]-\lambda\Psi_\nu^2\left[[g_\nu,\mathcal{K}_{\nu1}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}\right.\right.\\ \left.\left.+[g_\nu,\mathcal{K}_{\nu2}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\right]+[\Psi_\nu^2(d_\nu g_\nu),f_\nu^{(0)}]\\ -\lambda\left[[\Psi_\nu^2(d_\nu g_\nu),\mathcal{K}_{\nu1}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}\right.\\ \left.+[\Psi_\nu^2(d_\nu g_\nu),\mathcal{K}_{\nu2}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\Bigg\}.\end{aligned} \tag{32}$$

Since $\delta\Gamma_\nu$ is arbitrary in the internal region, the condition for the vanishing of $\delta_{\Gamma_\nu}(\delta^2H)_{\rm aux}$ is

$$\begin{aligned}d_\nu\left[\Psi_\nu^2[g_\nu,f_\nu^{(0)}]-\lambda\Psi_\nu^2\left[[g_\nu,\mathcal{K}_{\nu1}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}+[g_\nu,\mathcal{K}_{\nu2}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]\right]\\ +[\Psi_\nu^2(d_\nu g_\nu),f_\nu^{(0)}]-\lambda\left[[\Psi_\nu^2(d_\nu g_\nu),\mathcal{K}_{\nu1}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}\right.\\ \left.+[\Psi_\nu^2(d_\nu g_\nu),\mathcal{K}_{\nu2}]\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}\right]=0.\end{aligned} \tag{33}$$

As pointed out in Appendix B, it is not necessary to find the most general solution of this equation. It suffices to find solutions which are general enough to show that it is possible to make δ^2H negative in all cases of interest. With the expressions for $[g_\nu,\mathcal{H}_\nu]$ and $[g_\nu,f_\nu^{(0)}]$ found in Appendix B the perturbed energy, Eq. (24) becomes

$$\begin{aligned}\delta^2H=-\sum_\nu\int\frac{J(q_1,q_2)}{2m_\nu^3}\,\frac{\pi}{k_3}\,dq_1dq_2d^3p\Psi_\nu^2\,\frac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}\\ \times\mathcal{C}_{\nu a}\left[\mathcal{C}_{\nu a}+k_3\,\frac{\dfrac{\partial f_\nu^{(0)}}{\partial\mathcal{P}_{\nu3}}}{\dfrac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}}+\mathcal{C}_{\nu b}\,\frac{\dfrac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu1}}}{\dfrac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}}+\mathcal{C}_{\nu c}\,\frac{\dfrac{\partial f_\nu^{(0)}}{\partial\mathcal{K}_{\nu2}}}{\dfrac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}}\right].\end{aligned} \tag{34}$$

The electric-field energy term $1/8\pi\int d^3x\ \delta E^2$ has been dropped for the minimum of δ^2H, since the perturbed charge density can be made zero by an appropriate choice of the *signs* of Ψ_ν, which do not influence Eq. (34), and by making use of the freedom to choose $\delta\dot{\mathbf{A}}_{t=t_0}$, since this quantity is arbitrary. That the initial perturbed charge density can be made to vanish follows as in Refs. [7–9] (in a similar way, it can be shown that the initial current density perturbation $\delta\mathbf{j}_{t=t_0}$ can be made at least arbitrarily small).

As explained in Appendix B, $\mathcal{C}_{\nu a}$ is a completely arbitrary function of the constants of the motion $\mathcal{H}_\nu$, $\mathcal{P}_{\nu3}$, and $f_\nu^{(0)}$ for particles which do not have periodic orbits. For the exceptional case of particles with periodic orbits, $\mathcal{C}_{\nu a}$ is given by $\mathcal{C}_{\nu a}=(2\pi/\tau_\nu)n_{\nu0}$, where τ_ν is the period of the motion, and $n_{\nu0}$ is a completely arbitrary positive or negative integer.

The wave number corresponding to the symmetry direction k_3 is completely arbitrary, and $\mathcal{C}_{\nu b}$ and $\mathcal{C}_{\nu c}$ are arbitrary functions of the constants of the motion $\mathcal{H}_\nu$, $\mathcal{P}_{\nu3}$, $\mathcal{K}_{\nu1}$, and $\mathcal{K}_{\nu2}$.

VI. DISCUSSION

For the general symmetric equilibria considered here, it is easy to make the expression for the perturbation energy δ^2H negative by exploiting the fact that the functions $\Psi_\nu(\mathcal{H}_\nu,\mathcal{P}_{\nu3},\mathcal{K}_{\nu1},\mathcal{K}_{\nu2})$ and the constants of the motion $\mathcal{C}_{\nu a}(\mathcal{H}_\nu,\mathcal{P}_{\nu3},f_\nu^{(0)})$, $\mathcal{C}_{\nu b}(\mathcal{H}_\nu,\mathcal{P}_{\nu3},\mathcal{K}_{\nu1},\mathcal{K}_{\nu2})$, $\mathcal{C}_{\nu c}(\mathcal{H}_\nu,\mathcal{P}_{\nu3},\mathcal{K}_{\nu1},\mathcal{K}_{\nu2})$, and also k_3, can be arbitrarily chosen.

If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$ for some $\mathcal{H}_{\nu0}$, $\mathcal{P}_{\nu30}$, $\mathcal{K}_{\nu10}$, and $\mathcal{K}_{\nu20}$, δ^2H can easily be made negative. It suffices to localize Ψ_ν to the region in $\mathcal{H}_\nu$, $\mathcal{P}_{\nu3}$, $\mathcal{K}_{\nu1}$ and $\mathcal{K}_{\nu2}$ where $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu>0$. Outside this region, Ψ_ν vanishes. All other Ψ_μ are made equal to zero. One can then, for instance, choose k_3, $\mathcal{C}_{\nu b}$ and $\mathcal{C}_{\nu c}$ equal to zero. δ^2H is then negative for all $\mathcal{C}_{\nu a}\neq0$. Or, if $k_3\neq0$ is chosen, an appropriate choice of $\mathcal{C}_{\nu a}$ makes δ^2H negative, and so forth.

If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ for some $\mathcal{H}_{\nu0}$, $\mathcal{P}_{\nu30}$, $\mathcal{K}_{\nu10}$, and $\mathcal{K}_{\nu20}$, as is always the case, one localizes around these values in the way just explained. If $k_3\neq0$ and $\partial f_\nu^{(0)}/\partial\mathcal{P}_{\nu3}\neq0$, then, choosing, for instance, $\mathcal{C}_{\nu b}=\mathcal{C}_{\nu c}=0$, and $\mathcal{C}_{\nu a}(\mathcal{C}_{\nu a}+k_3[(\partial f_\nu^{(0)}/\partial\mathcal{P}_{\nu3})/(\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu)])<0$ yields $\delta^2H<0$. If, however, $k_3=0$ or $\partial f_\nu^{(0)}/\partial\mathcal{P}_{\nu3}=0$, but $\partial f_\nu^{(0)}/\partial\mathcal{K}_{\nu1}$ and $\partial f_\nu^{(0)}/\partial\mathcal{K}_{\nu2}$ are *not both zero*, then, choosing appropriate values for $\mathcal{C}_{\nu b}$ and $\mathcal{C}_{\nu c}$ yields $\delta^2H<0$.

If $\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu<0$ and $\partial f_\nu^{(0)}/\partial\mathcal{P}_{\nu3}=\partial f_\nu^{(0)}/\partial\mathcal{K}_{\nu1}=\partial f_\nu^{(0)}/\partial\mathcal{K}_{\nu2}=0$ for some $\mathcal{H}_{\nu0}$, $\mathcal{P}_{\nu30}$, $\mathcal{K}_{\nu10}$, and $\mathcal{K}_{\nu20}$, and all ν, then the distribution functions $f_\nu^{(0)}$ are *isotropic and monotonically decreasing* in this region of phase space, and it is not possible to make δ^2H negative by localizing the Ψ_ν's around these values. This is in agreement with previous results [7–9,12,13]. For the configurations considered here, however, this can be the case only in some regions of phase space because nonvanishing gradients with respect to $\mathcal{P}_{\nu3}$, $\mathcal{K}_{\nu1}$, and $\mathcal{K}_{\nu2}$ are necessary in order to produce the electric currents needed for equilibrim in the presence of pressure gradients.

VII. SUMMARY

The general expression for the perturbation energy of Maxwell-Vlasov equilibria was evaluated for symmetric configurations which have one ignorable variable (e.g., the toroidal angle φ in a tokamak or $u=m\varphi+kz$ in a helically symmetric configuration). Explicit dependence of the equilibrium distribution functions not only on the conserved particle energy $\mathcal{H}_\nu$, but also on the conserved momentum $\mathcal{P}_{\nu3}$

and on the two further constants of the motion $\mathcal{K}_{\nu 1}$ and $\mathcal{K}_{\nu 2}$, is essential for generating the electric currents necessary for equilibrium in the presence of pressure gradients. Owing to this dependence, the equilibrium distribution function of at least one particle species is anisotropic in **v** space.

Perturbations of negative energy ($\delta^2 H<0$) are easily obtained for any local deviation from monotonicity (i.e., if $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu>0$ for some $\mathcal{H}_{\nu 0}$, $\mathcal{P}_{\nu 30}$, $\mathcal{K}_{\nu 10,}$ and $\mathcal{K}_{\nu 20}$) of the distribution function of any of the particle species ν. But also if $\partial f_\nu^{(0)}/\partial \mathcal{H}_\nu<0$, it is possible to make $\delta^2 H$ negative because of the necessary anisotropy of the distribution function of at least one particle species (explicit dependence on $\mathcal{P}_{\nu 3}$, $\mathcal{K}_{\nu 1}$, and $\mathcal{K}_{\nu 2}$). No conditions are imposed on the wave numbers. In particular, large spacial gradients of the perturbations, and corresponding large perpendicular wave numbers, are *not* required. This enhances the relevance of these modes, which could be related to nonlinear instabilities and corresponding anomalous transport in tokamks and helically symmetric equilibria. In order to obtain these results, it was sufficient to consider perturbations which are initially electric neutral, and which satisfy $\delta \mathbf{B}_{t=t_0}=\mathbf{0}$.

APPENDIX A: PARTIAL DERIVATIVES AND POISSON BRACKETS

Let $q_i(\mathbf{x})$, $i=1,\ldots,3$ be generalized coordinates with covariant basis $\partial \mathbf{x}/\partial q_i$ and contravariant basis $\partial q_i/\partial \mathbf{x}=\boldsymbol{\nabla} q_i$. The corresponding covariant and contravariant velocity components are, respectively,

$$v_i(\mathbf{x},\mathbf{v})=\mathbf{v}\cdot \frac{\partial \mathbf{x}}{\partial q_i} \quad \text{and} \quad v^i(\mathbf{x},\mathbf{v})=\mathbf{v}\cdot \boldsymbol{\nabla} q_i=\dot{q}_i, \tag{A1}$$

and, correspondingly, for the components of the canonical momentum $\mathbf{p}_\nu=m_\nu \mathbf{v}+(e_\nu/c)\mathbf{A}^{(0)}$,

$$p_i=\mathbf{p}\cdot \frac{\partial \mathbf{x}}{\partial q_i}=m_\nu v_i+\frac{e_\nu}{c} A_i^{(0)}(\mathbf{x}), \tag{A2}$$

$$p^i=\mathbf{p}\cdot \boldsymbol{\nabla} q_i=m_\nu v^i+\frac{e_\nu}{c} A^{(0)i}(\mathbf{x}). \tag{A3}$$

This yields the relations

$$\left.\frac{\partial p_i}{\partial \mathbf{x}}\right|_{\mathbf{v}}=m_\nu \left.\frac{\partial v_i}{\partial \mathbf{x}}\right|_{\mathbf{v}}+\frac{e_\nu}{c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}, \quad \left.\frac{\partial p_i}{\partial \mathbf{p}}\right|_{\mathbf{x}}=\frac{\partial \mathbf{x}}{\partial q_i}, \quad \left.\frac{\partial p^i}{\partial \mathbf{p}}\right|_{\mathbf{x}}=\boldsymbol{\nabla} q_i. \tag{A4}$$

From the expression for $\mathbf{p}_\nu$, the following results are obtained:

$$\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}=m_\nu \left.\frac{\partial G_\nu}{\partial \mathbf{p}}\right|_{\mathbf{x}}, \tag{A5}$$

$$\begin{aligned}\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} &\equiv \boldsymbol{\nabla} q_i \left.\frac{\partial G_\nu}{\partial q_i}\right|_{\mathbf{v}}=\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}}+\frac{e_\nu}{c}\left[\frac{\partial \mathbf{A}^{(0)}}{\partial \mathbf{x}}\right]\cdot \left.\frac{\partial G_\nu}{\partial \mathbf{p}}\right|_{\mathbf{x}} \\ &=\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}}+\frac{e_\nu}{m_\nu c}\left[\frac{\partial \mathbf{A}^{(0)}}{\partial \mathbf{x}}\right]\cdot \left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}. \end{aligned}\tag{A6}$$

These relations yield

$$\begin{aligned}&\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}-\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} \\ &=m_\nu\left[\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}\left.\frac{\partial G_\nu}{\partial \mathbf{p}}\right|_{\mathbf{x}}-\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{p}}\right|_{\mathbf{x}}\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}}\right] \\ &\quad+\frac{e_\nu}{m_\nu c}\boldsymbol{\nabla} q_i\cdot\left[\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}\left[\frac{\partial \mathbf{A}^{(0)}}{\partial q_i}\cdot\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right]\right. \\ &\quad\left.-\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\left[\frac{\partial \mathbf{A}^{(0)}}{\partial q_i}\cdot\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right]\right] \\ &=m_\nu\left[\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}\left.\frac{\partial G_\nu}{\partial \mathbf{p}}\right|_{\mathbf{x}}-\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{p}}\right|_{\mathbf{x}}\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}}\right] \\ &\quad+\frac{e_\nu}{m_\nu c}\boldsymbol{\nabla} q_i\cdot\left[\frac{\partial \mathbf{A}^{(0)}}{\partial q_i}\times\left[\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}\times\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right]\right]. \end{aligned}\tag{A7}$$

Taking into account the generally valid relation $\boldsymbol{\nabla}\times\mathbf{A}^{(0)}=\boldsymbol{\nabla} q_i\times(\partial \mathbf{A}^{(0)}/\partial q_i)$, and the definition of the Poisson brackets

$$[f_\nu^{(0)},G_\nu]=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}\left.\frac{\partial G_\nu}{\partial \mathbf{p}}\right|_{\mathbf{x}}-\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{p}}\right|_{\mathbf{x}}\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}}, \tag{A8}$$

one then obtains

$$\begin{aligned}&\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}-\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} \\ &=m_\nu[f_\nu^{(0)},G_\nu]-\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\cdot\left[\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}\times\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}\right]. \end{aligned}\tag{A9}$$

The unperturbed Hamiltonian for particles of species ν is

$$\begin{aligned}H_\nu&=\frac{m_\nu}{2}\mathbf{v}^2+e_\nu\Phi^{(0)}(\mathbf{x}) \\ &=\frac{1}{2m_\nu}\left[\mathbf{p}_\nu-\frac{e_\nu}{c}\mathbf{A}^{(0)(\mathbf{x})}\right]^2+e_\nu\Phi^{(0)}(\mathbf{x}), \end{aligned}\tag{A10}$$

and, therefore,

$$\left.\frac{\partial H_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}}=m_\nu\mathbf{v}, \quad \left|\frac{\partial H_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}}=e_\nu\frac{\partial \Phi^{(0)}}{\partial \mathbf{x}}. \tag{A11}$$

With Eqs. (2), (3), and (A11) (with the appropriate substitutions made) taken into account, the time derivative of G_ν along unperturbed orbits $d_\nu G_\nu$ can be written as

$$d_\nu G_\nu = \mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\bigg|_{\mathbf{v}} + \frac{e_\nu}{m_\nu}\left[-\frac{\partial \Phi^{(0)}}{\partial \mathbf{x}} + \frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right]\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\bigg|_{\mathbf{x}}$$
$$= \frac{1}{m_\nu}\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\bigg|_{\mathbf{v}}\frac{\partial H_\nu}{\partial \mathbf{v}}\bigg|_{\mathbf{x}} - \frac{\partial H_\nu}{\partial \mathbf{x}}\bigg|_{\mathbf{v}}\frac{\partial G_\nu}{\partial \mathbf{v}}\bigg|_{\mathbf{x}}\right]$$
$$+ \frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\cdot\left[\frac{\partial G_\nu}{\partial \mathbf{v}}\bigg|_{\mathbf{x}} \times \frac{\partial H_\nu}{\partial \mathbf{v}}\bigg|_{\mathbf{x}}\right] = [G_\nu, H_\nu]. \tag{A12}$$

APPENDIX B: SOLUTION OF EULER'S EQUATION

The minimization of the perturbation energy $\delta^2 H$ with respect to $\Gamma_\nu(q_1,q_2,p_i)$, $i=1,\ldots,3$, yields Euler's equation, (33). In terms of the function $g_\nu = k_3 q_3 + \Gamma_\nu(q_1,q_2,p_i)$, this equation can be written as

$$d_\nu\Bigg\{2\Psi_\nu^2[g_\nu,f_\nu^{(0)}] + g_\nu[\Psi_\nu^2,f_\nu^{(0)}]$$
$$-2\lambda\Psi_\nu^2\left[[g_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu1}} + [g_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu2}}\right]$$
$$-\lambda g_\nu\left[[\Psi_\nu^2,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu1}} + [\Psi_\nu^2,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu2}}\right]\Bigg\}$$
$$-[g_\nu d_\nu(\Psi_\nu^2),f_\nu^{(0)}] + \lambda\Bigg[[g_\nu d_\nu(\Psi_\nu^2),\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu1}}$$
$$+[g_\nu d_\nu(\Psi_\nu^2),\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu2}}\Bigg] = 0, \tag{B1}$$

where, again,

$$d_\nu g_\nu = \left[\frac{dg_\nu}{dt}\right]_{\text{a.u.o.}} = [g_\nu, \mathcal{H}_\nu]. \tag{B2}$$

Here, the subscript a.u.o. means that the derivatives are taken along the unperturbed orbits of the particle motion in **x-p** space. Ψ_ν, which is the amplitude of the generating function G_ν, is an arbitrary, real, single-valued function. Based on previous experience [7–9,12], we choose *test functions* Ψ_ν which depend exclusively on constants of the motion,

$$\Psi_\nu = \Psi_\nu(\mathcal{H}_\nu, \mathcal{P}_{\nu3}, \mathcal{K}_{\nu1}, \mathcal{K}_{\nu2}), \tag{B3}$$

and Eq. (B1) reduces to

$$d_\nu\Bigg[2\Psi_\nu^2[g_\nu,f_\nu^{(0)}] - 2\lambda\Psi_\nu^2\Bigg[[g_\nu,\mathcal{K}_{\nu1}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu1}}$$
$$+[g_\nu,\mathcal{K}_{\nu2}]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu2}}\Bigg] + g_\nu[1-\lambda][\mathcal{K}_{\nu1},\mathcal{K}_{\nu2}]$$
$$\times\left[\frac{\partial \Psi_\nu^2}{\partial \mathcal{K}_{\nu1}}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu2}} - \frac{\partial \Psi_\nu^3}{\partial \mathcal{K}_{\nu2}}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu1}}\right]\Bigg] = 0. \tag{B4}$$

Therefore, the sum of all terms inside the brackets must be a constant of the motion. In this expression, the only term which depends explicitly on q_3 is $g_\nu = k_3 q_3 + \Gamma_\nu$, but not $[g_\nu, f_\nu^{(0)}]$ or $[g_\nu, \mathcal{K}_{\nu i}]$, $i=1$ and 2. Then, for general k_3, Ψ_ν and $[\mathcal{K}_{\nu1},\mathcal{K}_{\nu2}]$, this expression can be a constant of the motion only if

$$\lambda = 1, \tag{B5}$$

and Eq. (B4) reduces to

$$[d_\nu g_\nu, \mathcal{H}_\nu]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + [d_\nu g_\nu, \mathcal{P}_\nu]\frac{\partial f_\nu^{(0)}}{\partial \mathcal{P}_{\nu3}} = 0. \tag{B6}$$

It is not necessary to find the general solution to this equation. It suffices to find solutions which are general enough to show that it is possible to make $\delta^2 H$ negative in all cases of interest. With the ansatz

$$d_\nu g_\nu \equiv [g_\nu, \mathcal{H}_\nu] = \mathcal{C}_{\nu a}(\mathcal{H}_\nu, \mathcal{P}_{\nu3}, f_\nu^{(0)}(\mathcal{H}_\nu, \mathcal{P}_{\nu3}, \mathcal{K}_{\nu1}, \mathcal{K}_{\nu2})), \tag{B7}$$

Eq. (B6) is satisfied. $\mathcal{C}_{\nu a}$ is a completely arbitrary function of $\mathcal{H}_\nu$, $\mathcal{P}_{\nu3}$, and $f_\nu^{(0)}(\mathcal{H}_\nu, \mathcal{P}_{\nu3}, \mathcal{K}_{\nu1}$, and $\mathcal{K}_{\nu2})$ for particles which do not have periodic orbits. Explicitly, Eq. (B7) means

$$\left[\frac{d\Gamma_\nu}{dt}\right]_{\text{a.u.o.}} + k_3\dot{q}_3[q_1,q_2,p_i]$$
$$= \mathcal{C}_{\nu a}[\mathcal{H}_\nu, \mathcal{P}_{\nu3}, f_\nu^{(0)}(\mathcal{H}_\nu, \mathcal{P}_{\nu3}, \mathcal{K}_{\nu1}, \mathcal{K}_{\nu2})], \tag{B8}$$

an expression which allows, in principle, to determine Γ_ν, and thus g_ν, by integration along the unperturbed orbits under the constraint that the resulting generating function for the particle position and velocity perturbation $G_\nu(\mathbf{x},\mathbf{p})$ be single valued. This is similar to the case treated in Ref. [9], Appendix D. For the exceptional case of particles with periodic orbits, $\mathcal{C}_{\nu a}$ is obtained by integrating Eq. (B8) along the closed orbit in **x-p** space, and is then given by

$$\mathcal{C}_{\nu a} = \frac{2\pi}{\tau_\nu}\left[m_{\nu 0} + \frac{\tau_\nu}{2\pi}\langle k_3\dot{q}_3\rangle\right], \tag{B9}$$

where τ_ν is the period of the motion, the angles are the corresponding mean values along the unperturbed orbits, and $m_{\nu 0}$ is a completely arbitrary positive or negative integer. For these periodic orbits, $[q_3(t_0+\tau_\nu) - q_3(t_0)]k_3/(2\pi)$ is some integer number, which we call $m_{\nu q_3}$. Therefore

$$\mathcal{C}_{\nu a} = \frac{2\pi}{\tau_\nu}[m_{\nu 0} + m_{\nu q_3}] \equiv \frac{2\pi}{\tau_\nu} n_{\nu 0}. \tag{B10}$$

Since $m_{\nu 0}$ is a completely arbitrary integer, so is $n_{\nu 0}$.

For the evaluation of the perturbation energy, Eq. (24), not only $d_\nu g_\nu = [g_\nu, \mathcal{H}_\nu]$ is needed, but also $[g_\nu, f_\nu^{(0)}]$. The function g_ν itself, however, is not needed. Though this function is not a constant of the motion, $[g_\nu, f_\nu^{(0)}]$ is such a constant since

$$[[g_\nu, f_\nu^{(0)}], \mathcal{H}_\nu] = d_\nu[g_\nu, f_\nu^{(0)}] = [d_\nu g_\nu, f_\nu^{(0)}] = 0. \tag{B11}$$

Furthermore, one has

$$[g_\nu, f_\nu^{(0)}] = [g_\nu, \mathcal{H}_\nu] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + [g_\nu, \mathcal{P}_{\nu 3}] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{P}_{\nu 3}} + [g_\nu, \mathcal{K}_{\nu 1}] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 1}} + [g_\nu, \mathcal{K}_{\nu 2}] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 2}}$$
$$= \mathcal{C}_{\nu a} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + k_3 \frac{\partial f_\nu^{(0)}}{\partial \mathcal{P}_{\nu 3}} + [g_\nu, \mathcal{K}_{\nu 1}] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 1}} + [g_\nu, \mathcal{K}_{\nu 2}] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 2}}. \tag{B12}$$

The left-hand side and the first two terms of the right-hand side of this equation are constants of the motion. Therefore, the sum of the last two terms $[g_\nu, \mathcal{K}_{\nu 1}](\partial f_\nu^{(0)}/\partial \mathcal{K}_{\nu 1}) + [g_\nu, \mathcal{K}_{\nu 2}](\partial f_\nu^{(0)}/\partial \mathcal{K}_{\nu 2})$ must also be a constant of the motion. Explicitly, one has

$$d_\nu [g_\nu, \mathcal{K}_{\nu 1}] = [d_\nu g_\nu, \mathcal{K}_{\nu 1}] = [\mathcal{C}_{\nu a}(\mathcal{H}_\nu, \mathcal{P}_{\nu 3}, f_\nu^{(0)}), \mathcal{K}_{\nu 1}]$$
$$= [\mathcal{H}_\nu, \mathcal{K}_{\nu 1}] \frac{\partial \mathcal{C}_{\nu a}}{\partial \mathcal{H}_\nu} + [\mathcal{P}_{\nu 3}, \mathcal{K}_{\nu 1}] \frac{\partial \mathcal{C}_{\nu a}}{\partial \mathcal{P}_{\nu 3}} + [f_\nu^{(0)}, \mathcal{K}_{\nu 1}] \frac{\partial \mathcal{C}_{\nu a}}{\partial f_\nu^{(0)}}. \tag{B13}$$

$[\mathcal{H}_\nu, \mathcal{K}_{\nu 1}]$ vanishes because $\mathcal{K}_{\nu 1}$ is a constant of the motion, $[\mathcal{P}_{\nu 3}, \mathcal{K}_{\nu 1}]$ vanishes since $\mathcal{K}_{\nu 1}$ does not depend on q_3, and $[f_\nu^{(0)}, \mathcal{K}_{\nu 1}] = -[\mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}](\partial f_\nu^{(0)}/\partial \mathcal{K}_{\nu 2})$. Therefore

$$d_\nu [g_\nu, \mathcal{K}_{\nu 1}] = -[\mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}] \frac{\partial \mathcal{C}_{\nu a}}{\partial f_\nu^{(0)}} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 2}}. \tag{B14}$$

Integration of this equation along unperturbed orbits between times t_0 and t yields

$$[g_\nu, \mathcal{K}_{\nu 1}] = \mathcal{C}_{\nu b}(\mathcal{H}_\nu, \mathcal{P}_{\nu 3}, \mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}) - [\mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}] \frac{\partial \mathcal{C}_{\nu a}}{\partial f_\nu^{(0)}} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 2}} (t - t_0). \tag{B15}$$

In a similar way, one obtains

$$[g_\nu, \mathcal{K}_{\nu 2}] = \mathcal{C}_{\nu c}(\mathcal{H}_\nu, \mathcal{P}_{\nu 3}, \mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}) + [\mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}] \frac{\partial \mathcal{C}_{\nu a}}{\partial f_\nu^{(0)}} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 1}} (t - t_0) \tag{B16}$$

and, therefore,

$$[g_\nu, \mathcal{K}_{\nu 1}] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 1}} + [g_\nu, \mathcal{K}_{\nu 2}] \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 2}} = \mathcal{C}_{\nu b} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 1}} + \mathcal{C}_{\nu c} \frac{\partial f_\nu^{(0)}}{\partial \mathcal{K}_{\nu 2}}. \tag{B17}$$

Thus, although $[g_\nu, \mathcal{K}_{\nu 1}]$ and $[g_\nu, \mathcal{K}_{\nu 2}]$ are constants of the motion only if $[\mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}] = 0$, or if $f_\nu^{(0)}$ does not depend on $\mathcal{K}_{\nu 2}$ or $\mathcal{K}_{\nu 1}$, respectively, the combination of these two terms which appears in $\delta^2 H$ is a constant of the motion even if $[\mathcal{K}_{\nu 1}, \mathcal{K}_{\nu 2}] \neq 0$. The constants of the motion $\mathcal{C}_{\nu b}$ and $\mathcal{C}_{\nu c}$ *can be arbitrarily chosen*.

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